Final Exam Group Theory 2026-2027

Copied from the solution model, without solutions

1. Let G be the cyclic group of order 12 generated by g.
   (a) (9 Points) Find all subgroups of G (express their generators in terms of g).
   (b) (9 Points) Let H be the unique subgroup of size 2 of G. Find all subgroups of G/H. (you must express generators of each subgroup in terms of cosets of H)
2. (19 Points) Let G be a group of size 55, H be a group of size 20 and K be a group of size 11. It is given that there exists:
   - A nontrivial group homomorphism $\alpha : G \to H$ and
   - An onto group homomorphism $\beta : G \to K$.

   (a) (5 Points) Show that $\ker \alpha$ is a normal subgroup of G of size 11.
   (b) (4 Points) Show that $\ker \beta$ is a normal subgroup of G of size 5.
   (c) (10 Points) Show that G is cyclic.
3. (15 Points) True or false, justify your result briefly.
   (a) If G is a group such that every proper subgroup is abelian then G is abelian.
   (b) No group of order 33 is simple.
   (c) If G is free abelian group then G cannot contain any torsion element other than identity.
   (d) Let G be a group of size 14, then $1+1+1+1+1+2+7$ is a possible class equation for G.
   (e) Let H, K, and L be normal subgroups of G with $H < K < L$. Let $A = G/H$, $B = K/H$, and $C = L/H$. Then $(A/B)/(C/B)$ is isomorphic to G/L.
4. (18 Points) Let p be a prime and let G be a group of order p^n where $n \geq 1$. Let N be a normal subgroup of order p.
   Recall that the center of G is $Z(G) = \{z \in G | zg = gz \text{ for all } g \in G\}$.
   (a) (6 Points) Show that $g \cdot x = gxg^{-1}$ defines an action of G on the set N.
   (b) (6 Points) Show that every orbit of this action has exactly one element. Deduce that $N \subseteq Z(G)$.
   (c) (6 Points) Show that G has at least one normal subgroup of order p.
5. Let $G = S_5$, the symmetric group (permutation group) on $\{1, 2, 3, 4, 5\}$.
   (a) (5 Points) Show that any order 4 element of G is of the form $(a\ b\ c\ d)$ where a, b, c, d are pairwise distinct elements in $\{1, 2, 3, 4, 5\}$.
   (b) (4 Points) How many order 4 elements are there in G?
   (c) (5 Points) Let N be a normal subgroup of G containing an element of order 4. Using part (b), show that $|N| > 31$.
   (d) (6 Points) Show that $|N| > 60$ and deduce that $N = G$. (*Hint:* Can a product of two 4-cycles be a 3-cycle?)

BONUS (5 Points) Write your favorite result from this class, you do not need to prove it.